REVIEW

of the official reviewer for dissertation work

of Bayan Bekbolat on the theme «Dunkl analysis and application to inverse source problems»
presented for the degree of Doctor of Philosophy (PhD) in the specialty «6D060100 - Mathematics».

№	Criteria	Eligibility (one of the options must be checked)	Justification of the position of the official reviewer
1.	The topic of the thesis (as of the date of its approval) corresponds to the directions of development of science and/or state programs	1.1 Compliance with priority areas of science development or government programs: 1) The thesis was completed within the framework of a project or target program financed from the state budget (indicate the name and number of the project or program) 2) The thesis was completed within the framework of another state program (indicate the name of the program) 3) The dissertation corresponds to the priority direction of the development of science, approved by the Higher Scientific and Technical Commission under the Government of the Republic of Kazakhstan (indicate the direction)	The dissertation work was carried out within the framework of the grant project of Committee of Science of the MSHE RK AP14972634 «Inverse source problems for pseudo-parabolic equations associated with Dunkl and Jacobi operators» («Zhas Galym», 2022 – 2024). It corresponds to the direction of fundamental and applied research in the field of mathematics and mechanics.
2.	Importance for science	The work makes/does not make a significant contribution to science, and its importance is well disclosed/not disclosed	Currently, the solution of direct and inverse problems for partial differential equations are being intensively studied. Some equations used in stochastic analysis include the Dunkl operators. Such equations are given in an infinite set, and the intermediate and minor coefficients are singular. Traditional methods are not suitable for solving them. The dissertation

3.	The principle of independence	Self-reliance level: 1) High; 2) Medium; 3) Low; 4) No independence	work is devoted to the creation and use of the technics for the study of such equations. From this we can see that the research work is relevant. It makes a significant contribution to science. The scientific significance is clearly stated in the text of the work. The level of writing itself is high. The work is written in a unique style, the justifications are clear, and the conclusions are presented with full arguments.
4.	The principle of inner unity	4.1 Justification of the relevance of the thesis: 1) Justified; 2) Partially justified; 3) Not justified. 4.2 The content of the thesis reflects the topic of the thesis: 1) Reflects; 2) Partially reflects; 3) Does not reflect	The relevance and significance of the dissertation is fully disclosed and justified in the introduction. The content of the dissertation fully defines the topic of the dissertation. The main part of the dissertation consists of four parts. The first introductory part summarizes the previous developments and achievements of Dunkl analysis. In the second part, the applicant gives a detailed overview of the concepts and methods from Dunkl analysis necessary for his analysis. In the third part, he presented results obtained for pseudo-differential operators generated by the Dunkl operator. In the fourth part, he studied three

		different inverse source problems for the heat and pseudo-parabolic equations generated by the Dunkl operators.
	4.3. The purpose and objectives correspond to the topic of the thesis: 1) correspond; 2) partially correspond; 3) do not correspond	The purpose and objectives of the work are to complement and apply Dunkl's analysis, which means that they fully correspond to the topic of the dissertation.
	4.4 All sections and provisions of the thesis are logically interconnected: 1) completely interconnected; 2) the interconnection is partial; 3) there is no interconnection	All sections of the dissertation are logically fully connected. The results of the second and third sections are used in the fourth section.
	4.5 The new solutions (principles, methods) proposed by the author are reasoned and evaluated in comparison with the known solutions: 1) there is a critical analysis; 2) partial analysis; 3) the analysis does not represent one's own opinions, but quotes from other authors	The author significantly develops Dunkl analysis and, using it, to solve several inverse source problems for new classes of differential equations. Critical analysis and comparison are given in the introduction of the work and at the beginning of the fourth part.
5.	Scientific novelty principle	5.1 Are the scientific results and provisions new? 1) completely new; 2) partially new (25-75% are new); 3) not new (less than 25% are new)	In the dissertation, the applicant showed: that the pseudo-differential, amplitude, adjoint and transpose operators, gerated by the Dunkl operators are linear continuous operators on Schwartz classes; Derived an estimate to the kernel of the pseudo-differential operator generated by the Dunkl operator; showed that the solutions of three different inverse source problems for heat and pseudo-parabolic equations

№	Question	Response
	5.2 Are the dissertation findings new? 1) completely new; 2) partially new (25-75% are new); 3) not new (less than 25% are new)	generated by the Dunkl operators are exist and unique. All these scientific results are new and, in general, significantly develop operator theories and partial differential equations. The conclusion of the dissertation is based on the results obtained in it and is therefore completely new.
	5.3 Technical, technological, economic or management decisions are new and reasonable: 1) completely new; 2) partially new (25-75% are new); 3) not new (less than 25% are new)	The results obtained are theoretical in nature. In addition, they can be used in creating algorithms for solving inverse problems for partial differential equations of variable coefficients, as well as in teaching special courses in mathematics in higher educational institutions.
6.	The validity of the main findings	All main conclusions are based on scientifically significant evidence or well-grounded (for qualitative research and areas of training in the arts and humanities)	All conclusions obtained in the dissertation are mathematically proven and published in the materials of international conferences and scientific journals.
7.	The main provisions for the defense	It is necessary to answer the following questions for each provision separately: 7.1 Is the provision proven? 1) proven; 2) rather proven; 3) rather not proven; 4) not proven 7.2 Is it trivial? 1) yes; 2) no 7.3 Is it new?	7.1 The results of the dissertation are fully proven. 7.2 The main results of the dissertation are not trivial. 7.3 The main results presented for defense are new. In general, they are a significant contribution to the Dunkl analysis and the theory of inverse source problems for partial differential equations involving the

	1) <u>yes</u>; 2) no 7.4 Application level: 1) narrow; 2) medium; 3) <u>wide</u> 7.5 Is it proven in the article? 1) <u>yes</u>; 2) no	Dunkl operators. 7.4 Inverse source problems for equations involving the Dunkl operator are used in modeling various processes of stochastic analysis. Therefore, the level of application of the obtained scientific results is wide. 7.5 Part of the main results are proven in the applicant's paper «Direct and inverse problems for time fractional heat equation generated by Dunkl operator». Paper published in the journal «Journal of Inverse and Ill-Posed Problems (31:3 (2023), 393-408) » with non-zero impact factor, which included in the Web of Sciences database. Other results are also published.
8.	The principle of reliability Reliability of sources and information provided 8.1 Choice of methodology - is justified or the methodology is described in sufficient detail 1) <u>yes</u>; 2) no	In the third part of the dissertation, he used the methods of the Dunkl analysis and theory of classical pseudo-differential operators, and in the fourth part he used the version of the classical Fourier method adapted for the Dunkl analysis. The use of these methods is justified by a new scientific results. To be precise, by using a form of the classical Fourier method adapted to Dunkl analysis, he receives an explicit solution to the inverse source problems.

#	Principle	Criterion	Description
		8.2 The results of the thesis were obtained using modern methods of scientific research and methods of processing and interpreting data using computer technologies: 1) yes; 2) no	The applicant gave a specific example of conclusions in the fourth part. It used new computer programs to calculate complex integrals and to show the dynamics of the solution, resulting in a clear solution and images.
		8.3 Theoretical conclusions, models, identified relationships and patterns have been proven and confirmed by experimental research (for areas of training in pedagogical sciences, the results have been proven on the basis of a pedagogical experiment): 1) yes; 2) no	Theoretical conclusions of the dissertation do not require experimental research. The constructiveness of the created and used methods justifies the reliability of the research. Theorems in the dissertation are presented mathematically competent and with complete proofs.
		8.4 Important statements are confirmed / partially confirmed / not confirmed by references to current and reliable scientific literature	Important conclusions are supported by references to accurate and reliable scientific literature.
		8.5 Used literature sources are sufficient/not sufficient for a literature review	The list of references consists of 99 works and is sufficient for a literary review.
9	Practical value principle	9.1 The thesis has theoretical value: 1) yes; 2) no	There is a theoretical importance of the obtained results. It is a significant contribution to Dunkl's analysis and to the theory of inverse source problems for partial differential equations in which the Dunkl operator participate.
		9.2 The thesis is of practical importance and there is a high probability of applying the results obtained in practice: 1) yes; 2) no	The dissertation has practical significance: its results can be used in stochastic analysis.

	9.3 Are the practice suggestions new? 1) <u>completely new</u>; 2) partially new (25-75% are new); 3) not new (less than 25% are new)		The Dunkl operator brings new and important problems of stochastic analysis to the partial differential equations. Therefore, the practical use of the results are completely new.
10.	The quality of writing and design	Academic writing quality: 1) <u>high</u>; 2) average; 3) below average; 4) low.	The dissertation is designed mathematically competently in accordance with the requirements, the quality of academic writing is high.

Conclusion. I believe that for his significant contribution to the Dunkl analysis and theory of partial differential equations with the Dunkl operator, applicant Bayan Bekbolat is worthy of being awarded the degree of Doctor of Philosophy in the specialty "6D060100 – Mathematics".

Official Reviewer:
Professor of L.N. Gumilyov Eurasian National
University, d.ph.-m.s.
(place of work, academic title)

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(signature)

Ospanov K.N.
(FULL NAME)